JOURNAL OF APPROXIMATION THEORY 27, 197–202 (1979)

Korovkin's Theorems for C^*-Algebras

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Communicated by Oved Shisha

Received April 20, 1978

1. The well-known theorems of Korovkin [5] assert the following:

(i) If $\{f_n\}$ is a sequence of positive linear functionals on $C[0, 1]$ and if t_0 is a point in $[0, 1]$, then $f_n(x) \to x(t_0)$ for all $x \in C[0, 1]$, provided only that $f_n(1) \to 1$ and $f_n((t - t_0)^2) \to 0$.

(ii) If $\{T_n\}$ is a sequence of positive linear operators on $C[0, 1]$ into itself, then $T_n(x) \to x$, in the uniform topology, provided only that $T_n(1) \to 1$, $T_n(t) \to t$, and $T_n(t^2) \to t^2$ in that topology.

Here $C[0, 1]$ denotes the algebra of all continuous real-valued functions on the unit interval $[0, 1]$. These theorems are fundamental in Korovkin's theory of approximation. Indeed, in [5], several proofs have been given to the Weierstrass approximation theorem for algebraic polynomials, all based on (ii).

In the present paper, we generalize (i) and (ii) to arbitrary C^*-algebras. Our main results are Theorems 2.2 and 3.4, corresponding, respectively, to (i) and (ii). We note that Arveson's method in [2] plays an essential role in the proof of Theorem 3.4.

Korovkin-type theorems for noncommutative C^*-algebras have been proved in [7] and elsewhere.

2. Throughout the paper, let A be a C^*-algebra with an identity and A^{**} be the second dual of A. Then A^{**} is a W^*-algebra (von Neumann algebra). We shall consider A as lying in A^{**} under the canonical embedding. In order to obtain an abstract version of Korovkin's result (i), we will use the following notation.

DEFINITION 2.1. Let f be a pure state of A and N_f be the left kernel of f, that is, the set of all elements x in A with $f(x^*x) = 0$. A positive element x in A is said to peak for f if $E(x) = 1 - E(f)$, where $E(x)$ and $E(f)$ denote, respectively, the supports of x and f in A^{**}.

0021-9045/79/110197–06$02.00/0

THEOREM 2.2. *Let f be a pure state of A and let x peak for f. If a net of positive linear functionals f_λ on A satisfies the two conditions* $\lim f_\lambda(1) = 1$ *and* $\lim f_\lambda(x) = 0$, *then* $\{f_\lambda\}$ *converges weak* to* f: $\lim f_\lambda(z) = f(z)$ *for all* $z \in A$.

Proof. Set $L = \{a \in A: \lim f_\lambda(a^*a) = 0\}$. Then L is a closed left ideal of A. We first show that $L = N_f$. In fact, let g be a pure state of A with $L \subset N_g$. Since $f_\lambda(x^2) \leqslant \| x \| f_\lambda(x)$ for all λ, we have $x \in L$ by the assumption $\lim f_\lambda(x) = 0$. Then $x \in N_g$ and hence $E(g) \leqslant 1 - E(x) = E(f)$. Since $E(f)$, $E(g)$ are minimal in A^{**}, we have $E(f) = E(g)$ and so $f = g$. It follows by Théorème 2.9.5(iii) in [4] that $L = N_f$. Now, let $y \in \mathrm{Ker}(f)$. Then there exist elements a, b in N_f with $y = a + b^*$. Observe that $|f_\lambda(y)|^2 \leqslant 2 \|f_\lambda\| f_\lambda(a^*a + b^*b)$ for all λ. Then $\lim f_\lambda(y) = 0$ because $a, b \in L$ and $\lim \|f_\lambda\| = 1$. On the other hand, each $z \in A$ can be expressed as $y + \alpha \cdot 1$ with some $y \in \mathrm{Ker}(f)$ and a certain complex number α. Therefore we have

$$\lim f_\lambda(z) = \lim f_\lambda(y) + \alpha \lim f_\lambda(1) = \alpha = f(z)$$

for all $z \in A$ and the proof is complete.

Observe that if A is commutative and a is an element of A such that the closed ideal M generated by a is maximal, then $| a |$ peaks for the character defined by M. Therefore the following result established by Choda and Echigo [3] is a special case of Theorem 2.2.

COROLLARY 2.3 (Choda and Echigo [3]). *Let A be a commutative C^*-algebra with identity, M be a principal maximal ideal generated by a and χ the character defined by M. Let $\{f_n\}$ be a sequence of positive linear functionals on A such that* $\lim f_n(1) = 1$ *and* $\lim f_n(| a |^2) = 0$. *Then* $\lim f_n(x) = \chi(x)$ *for all* $x \in A$.

We next show that if A is separable, then, for an arbitrary pure state f of A, there always exists an element which peaks for f. To see this, we first give a characterization of peaking elements.

LEMMA 2.4. *Let f be a pure state of A and x be a positive element of N_f. Then, in order for x to peak for f it is necessary and sufficient that each pure state g of A with $g(x) = 0$ is equal to f.*

Proof. If x peaks for f and g is a pure state of A with $g(x) = 0$, then $E(g) \leqslant 1 - E(x) = E(f)$. Since $E(f)$ and $E(g)$ are minimal, $f = g$ and the necessity is proved. Now suppose that each pure state g of A with $g(x) = 0$ is equal to f. Set $p = 1 - E(x)$ and $L = \{a \in A: ap = 0\}$. Then L is a closed left ideal of A which contains x. Therefore if g is a pure state of A with $L \subset N_g$, then $g(x) = 0$, so that $g = f$ by the assumption. In other words,

$L = N_f$. Note that $N_f = \{a \in A: aE(f) = 0\}$. Hence $p = E(f)$ because p and $E(f)$ are closed in A^{**} (cf. [1, p. 279]), and the sufficiency is proved.

THEOREM 2.5. *Let A be a separable C^*-algebra with an identity and f be a pure state of A. Then there exists an element which peaks for f.*

Proof. Since $N_f \cap N_f^*$ is a separable C^*-algebra, it has an approximate identity $\{e_1, e_2, ...\}$. Then $\lim e_n = 1 - E(f)$ in the weak* topology of A^{**}. Set

$$x = \sum_{n=1}^{\infty} 2^{-n} e_n .$$

We show that x peaks for f. Obviously, $x \in N_f$. If g is a pure state of A with $g(x) = 0$, then $g(e_n) = 0$ for all $n = 1, 2, ...$. Therefore $g(1 - E(f)) = 0$ and hence $E(g) \leqslant E(f)$. Since $E(f)$ and $E(g)$ are minimal in A^{**}, we have $g = f$. It follows by Lemma 2.4 that x peaks for f.

3. In this section, we shall generalize Korovkin's result (ii) for positive linear operators to arbitrary C^*-algebras. To this end, we will use the following notation.

DEFINITION 3.1. Let K be a subset of A which contains the identity, and let $E(A)$ be the set of all states of A. The set of all f in $E(A)$ such that f is the only positive linear functional on A which extends $f|_K$ is called the Choquet boundary of $E(A)$ for K, and is denoted by $\partial_K(E(A))$.

Let A_{sa} be the real linear space consisting of all self-adjoint elements of A. We then have the following

LEMMA 3.2. *Let S be a self-adjoint linear subspace of A containing the identity. Then for each $a \in A_{sa}$ and $f \in \partial_S(E(A))$, $f(a) = \inf\{f(x): x \in S, x \geqslant a\}$.*

Proof. Denote this inf by m. By the definition of m, $f(a) \leqslant m$. We further have that $f(a) = m$ if $a \in S \cap A_{sa}$. To complete the proof, we need to show that $f(a) = m$ if $a \notin S \cap A_{sa}$. Let $\mathbb{R}$ be the field of all real numbers and set

$$L = \{x + \alpha a: x \in S \cap A_{sa}, \alpha \in \mathbb{R}\}.$$

Then L is a real linear subspace of A_{sa}. For each $\alpha \in \mathbb{R}$ and $x \in S \cap A_{sa}$, let $g_0(x + \alpha a) = f(x) + \alpha m$. Then g_0 is a real linear functional on L. We show $g_0(y) \geqslant 0$ for each positive element y in L. Let $y = x + \alpha a$ be a positive element in L, where $x \in S \cap A_{sa}$, $\alpha \in \mathbb{R}$. If $\alpha \geqslant 0$, then $f(\alpha a) = \alpha f(a) \leqslant \alpha m$ and so $g_0(y) = f(x) + \alpha m \geqslant f(x) + f(\alpha a) = f(y) \geqslant 0$. If $\alpha < 0$, then we get $a \leqslant -\alpha^{-1}x \in S \cap A_{sa}$, so that $m \leqslant f(-\alpha^{-1}x)$ and hence

$g_0(y) = f(x) + \alpha m \geqslant 0$, as required. It follows by Krein's extension theorem (see [8, p. 227]) that there exists a linear functional g_1 on A_{sa} such that $g_1(x) \geqslant 0$ for all positive elements x in A and $g_0 = g_1 \mid L$. For each $x \in A$, set

$$g(x) = g_1((x + x^*)/2) + ig_1((x - x^*)/2i),$$

where $i = (-1)^{1/2}$. Then g is a positive linear functional on A such that $g \mid S = f \mid S$. This implies $g = f$, since $f \in \partial_S(E(A))$. We therefore have $m = g_0(a) = g_1(a) = g(a) = f(a)$ and the proof is complete.

Let K be a subset of A and F be a subset of $E(A)$. For every $a, b \in A_{sa}$ and $\epsilon > 0$, we now set

$$K(a) = \{x \in K: x \geqslant a, x \in A_{sa}\},$$
$$F(b; a, \epsilon) = \{f \in F: f(b) < f(a) + \epsilon\}.$$

We then have the following

LEMMA 3.3. *Let S be as in Lemma 3.2 and let F be a weak*-closed subset of $E(A)$ such that $F \subset \partial_S(E(A))$. If $a \in A_{sa}$ and $\epsilon > 0$, then there exists a finite subset $\{x_1, \ldots, x_n\}$ of $S(a)$ such that $F = F(x_1; a, \epsilon) \cup \cdots \cup F(x_n; a, \epsilon)$.*

Proof. Let $a \in A_{sa}$ and $\epsilon > 0$. We first show that $\bigcap \{F \backslash F(x; a, \epsilon): x \in S(a)\} = \varnothing$. Indeed, if there exists an element f_0 in $\bigcap \{F \backslash F(x; a, \epsilon): x \in S(a)\}$, then $f_0(x) \geqslant f_0(a) + \epsilon$ for all $x \in S(a)$. It follows by Lemma 3.2 that $f_0(a) = \inf\{f_0(x): x \in S(a)\} \geqslant f_0(a) + \epsilon$. We thus get $\epsilon \leqslant 0$, contrary to the hypothesis $\epsilon > 0$. In other words, $F = \bigcup \{F(x; a, \epsilon): x \in S(a)\}$. Since F is weak*-closed and each $F(x; a, \epsilon)$ is weak*-open relative to F, there exists a subset $\{x_1, \ldots, x_n\}$ of $S(a)$ such that $F = F(x_1; a, \epsilon) \cup \cdots \cup F(x_n; a, \epsilon)$ and the proof is complete.

Let $P(A)$ be the set of all pure states of A and $\bar{P}(A)$ be the pure state space of A, that is, the closure of $P(A)$ in the weak* topology in A^*. The following result is our promised generalization of Korovkin's theorem (ii).

THEOREM 3.4. *Let K be a subset of A containing the identity such that $\bar{P}(A) \subset \partial_K(E(A))$. If $\{T_\lambda\}$ is a net of positive linear operators on A into itself such that $\lim \| T_\lambda(x) - x \| = 0$ for all $x \in K$, then $\lim \| T_\lambda(x) - x \| = 0$ for all $x \in A$.*

Proof. To prove the theorem, we need only show that $\lim \| T_\lambda(a) - a \| = 0$ for all $a \in A_{sa}$. Let a be any fixed element of A_{sa} and let $\epsilon > 0$. Let S be the linear span of $K \cup K^*$, where $K^* = \{x^*: x \in K\}$. Then S is a self-adjoint subspace of A containing the identity and $\bar{P}(A) \subset \partial_S(E(A))$. It follows by Lemma 3.3 that there exists a finite subset $\{x_1, \ldots, x_n\}$ of $S(a)$ such that

$\bar{P}(A) = \bar{P}(A)(x_1 ; a, \epsilon) \cup \cdots \cup \bar{P}(A)(x_n ; a, \epsilon)$. Note that $\lim \| T_\lambda(x) - x \| = 0$ for all $x \in S$ and, hence, there exists λ'_ϵ such that $\max\{\| T_\lambda(x_j) - x_j \|: 1 \leqslant j \leqslant n\} < \epsilon$ for all $\lambda \geqslant \lambda'_\epsilon$. Then, for each j, we have

$$\begin{aligned} T_\lambda(a) &\leqslant T_\lambda(x_j) = x_j + T_\lambda(x_j) - x_j \\ &\leqslant x_j + \| T_\lambda(x_j) - x_j \| \cdot 1 \\ &\leqslant x_j + \epsilon \cdot 1 \end{aligned}$$

for all $\lambda \geqslant \lambda'_\epsilon$. Choose $g \in P(A)$. There exists an x_k such that $g \in \bar{P}(A)(x_k ; a, \epsilon)$; hence we have

$$\begin{aligned} g(T_\lambda(a)) &\leqslant g(x_k + \epsilon \cdot 1) = g(x_k) + \epsilon \\ &\leqslant g(a) + 2\epsilon = g(a + 2\epsilon \cdot 1) \end{aligned}$$

for all $\lambda \geqslant \lambda'_\epsilon$. It follows that $T_\lambda(a) \leqslant a + 2\epsilon \cdot 1$ for all $\lambda \geqslant \lambda'_\epsilon$. Similarly, there exists λ''_ϵ such that $T_\lambda(-a) \leqslant -a + 2\epsilon \cdot 1$ for all $\lambda \geqslant \lambda''_\epsilon$. Choose an index λ_ϵ such that $\lambda_\epsilon \geqslant \lambda'_\epsilon$ and $\lambda_\epsilon \geqslant \lambda''_\epsilon$. Then, for every $\lambda \geqslant \lambda_\epsilon$, we have $-2\epsilon \cdot 1 \leqslant T_\lambda(a) - a \leqslant 2\epsilon \cdot 1$ and, hence, $\| T_\lambda(a) - a \| \leqslant 2\epsilon$. Since ϵ is arbitrary, $\lim \| T_\lambda(a) - a \| = 0$ for all $a \in A_{sa}$ and the proof is complete.

COROLLARY 3.5 (Nakamoto and Nakamura [6]). *Suppose A is commutative and every maximal ideal of A is principal. Let $a_1, \ldots, a_k$ be elements of A having the following property: For each maximal ideal M of A, there exists an element m of A such that m generates M and $| m |^2$ is expressible as a linear combination of $a_1, \ldots, a_k$. If $\{T_n\}$ is a sequence of positive linear operators on A into itself such that* $\lim \| T_n(a_j) - a_j \| = 0$ *for $j = 0, 1, \ldots, k$, where $a_0 = 1$, then* $\lim \| T_n(a) - a \| = 0$ *for all $a \in A$.*

Proof. Set $K = \{a_0, a_1, \ldots, a_k\}$. Obviously, $P(A)$ is weak*-closed in $E(A)$. So, by Theorem 3.4, we need only show that $P(A) \subset \partial_K(E(A))$. Let $f \in P(A)$ and $M = \operatorname{Ker}(f)$. Since M is a maximal ideal of A, M is generated by some element m of A. Note that $f(| m |) = 0$. If g is a state of A with $f \mid K = g \mid K$, then by the property of $\{a_1, \ldots, a_k\}$, $g(| m |^2) = f(| m |^2)$ and so $g(| m |^2) = 0$. It follows by Corollary 2.3 that $g = f$. We thus get $f \in \partial_K(E(A))$ and the proof is complete.

ACKNOWLEDGMENTS

We thank the referee and Professor O. Shisha for suggestions for improving the paper and Professor M. Hasumi for helpful discussions.

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